Distributionally Robust Stochastic and Online Optimization Driven by Data/Samples

Models/Algorithms for Learning and Decision Making

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November 14, 2018
(Joint work with many others ...)

Outline

- Introduction to Distributionally Robust Optimization (DRO)
- DRO under Moment, Likelihood and Wasserstein Bounds
- Price of Correlation of High-Dimension Uncertainty
- MDP Value-Iteration Sample/Computation Complexities
- Online Linear Optimization and Dynamic Resource Allocation

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Develop tractable and provable models and algorithms for optimization with uncertain and online data.

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Introduction to DRO

We start from considering a stochastic optimization problem as follows:

$$\text{maximize}_{\mathbf{x} \in X} \quad \mathbb{E}_{F_\xi}[h(\mathbf{x}, \xi)] \tag{1}$$

where $\mathbf{x}$ is the decision variable with feasible region X, ξ represents random variables satisfying joint distribution F_ξ.

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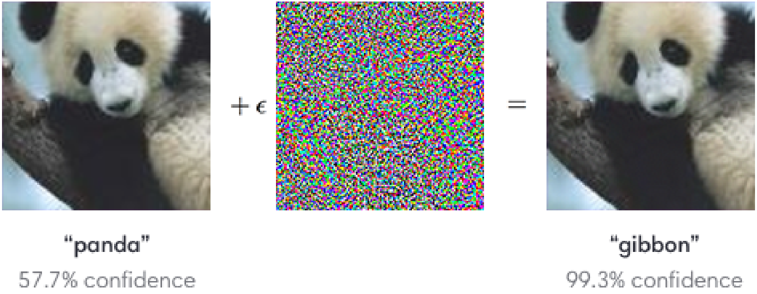
where $\mathbf{x}$ is the decision variable with feasible region X, ξ represents random variables satisfying joint distribution F_ξ.

- Pros: In many cases, the expected value is a good measure of performance
- Cons: One has to know the exact distribution of ξ to perform the stochastic optimization. Deviant from the assumed distribution may result in sub-optimal solutions. Even know the distribution, the solution/decision is generically risky.

Learning with Noises



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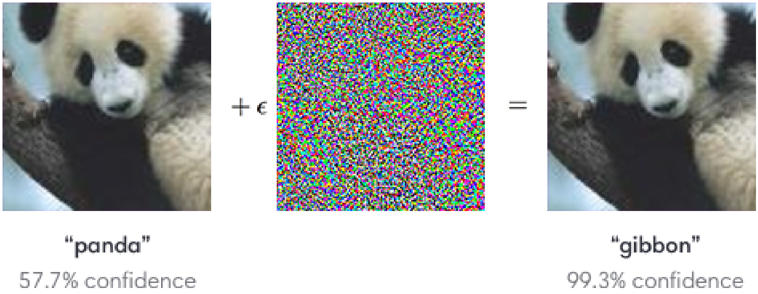

Goodfellow et al. [2014]

Robust Optimization

In order to overcome the lack of knowledge on the distribution, people proposed the following (static) robust optimization approach:

$$\text{maximize}_{\mathbf{x} \in X} \quad \min_{\xi \in \Xi} h(\mathbf{x}, \xi) \tag{2}$$

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- Pros: Robust to any distribution; only the support of the parameters are needed.
- Cons: Too conservative. The decision that maximizes the worst-case pay-off may perform badly in usual cases; e.g., Ben-Tal and Nemirovski [1998, 2000], etc.

Motivation for a Middle Ground

- In practice, although the exact distribution of the random variables may not be known, people usually know certain observed samples or training data and other statistical information.

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- In practice, although the exact distribution of the random variables may not be known, people usually know certain observed samples or training data and other statistical information.

- Thus we could choose an intermediate approach between stochastic optimization, which has no robustness in the error of distribution; and the robust optimization, which admits vast unrealistic single-point distribution on the support set of random variables.

Distributionally Robust Optimization

A solution to the above-mentioned question is to take the following Distributionally Robust Optimization/Learning (DRO) model:

$$\text{maximize}_{\mathbf{x}\in X} \quad \min_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi}[h(\mathbf{x},\xi)] \tag{3}$$

In DRO, we consider a set of distributions $\mathcal{D}$ and choose one to maximize the expected value for any given $\mathbf{x} \in X$.

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When choosing $\mathcal{D}$, we need to consider the following:

- Tractability
- Practical (Statistical) Meanings
- Performance (the potential loss comparing to the benchmark cases)

Sample History of DRO

- First introduced by Scarf [1958] in the context of inventory control problem with a single random demand variable.
- Distribution set based on moments: Dupacova [1987], Prekopa [1995], Bertsimas and Popescu [2005], Delage and Y [2007,2010], etc
- Distribution set based on Likelihood/Divergences: Nilim and El Ghaoui [2005], Iyanger [2005], Wang, Glynn and Y [2012], etc
- Distribution set based on Wasserstein ambiguity set: Mohajerin Esfahani and Kuhn [2015], Blanchet, Kang, Murthy [2016], Duchi, Glynn, Namkoong [2016]
- Axiomatic motivation for DRO: Delage et al. [2017]; Ambiguous Joint Chance Constraints Under Mean and Dispersion Information: Hanasusanto et al. [2017]
- Lagoa and Barmish [2002] and Shapiro [2006] simply considers a set containing unimodal distributions, Kleinberg et al. [1997] and M'ohring et al. [1999] considers the product distribution

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DRO with Moment Bounds

Define

$$\mathcal{D} = \left\{ F_\xi \left| \begin{array}{l} P(\xi \in \Xi) = 1 \\ (\mathbb{E}[\xi] - \mu_0)^T \Sigma_0^{-1} (\mathbb{E}[\xi] - \mu_0) \leq \gamma_1 \\ \mathbb{E}[(\xi - \mu_0)(\xi - \mu_0)^T] \preceq \gamma_2 \Sigma_0 \end{array} \right. \right\}$$

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Theorem

Under mild technical conditions, the DRO model can be solved to any precision ϵ in time polynomial in $\log(1/\epsilon)$ and the sizes of $\mathbf{x}$ and ξ

Delage and Y [2010]

Confidence Region on F_ξ

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Theorem

Consider

$$D(\gamma_1, \gamma_2) = \left\{ F_\xi \left| \begin{array}{l} P(\xi \in \Xi) = 1 \\ (\mathbb{E}[\xi] - \mu_0)^T \Sigma_0^{-1} (\mathbb{E}[\xi] - \mu_0) \leq \gamma_1 \\ \mathbb{E}[(\xi - \mu_0)(\xi - \mu_0)^T] \preceq \gamma_2 \Sigma_0 \end{array} \right. \right\}$$

where μ_0 and Σ_0 are point estimates from the empirical data (of size m) and Ξ lies in a ball of radius R such that $||\xi||_2 \leq R$ a.s..

Then for $\gamma_1 = O(\frac{R^2}{m} \log(4/\delta))$ and $\gamma_2 = O(\frac{R^2}{\sqrt{m}} \sqrt{\log(4/\delta)})$,

$$P(F_\xi \in D(\gamma_1, \gamma_2)) \geq 1 - \delta$$

DRO with Likelihood Bounds

Define the distribution set by the constraint on the likelihood ratio. With observed Data: $\xi_1, \xi_2, ...\xi_N$, we define

$$\mathcal{D}_N = \left\{ F_\xi \middle| \begin{array}{l} P(\xi \in \Xi) = 1 \\ L(\xi, F_\xi) \geq \gamma \end{array} \right\}$$

where γ adjusts the level of robustness and N represents the sample size.

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For example, assume the support of the uncertainty is finite

$$\xi_1, \xi_2, ...\xi_n$$

and we observed m_i samples on ξ_i. Then, F_ξ has a finite discrete distribution $p_1, ..., p_n$ and

$$L(\xi, F_\xi) = \sum_{i=1}^{n} m_i \log p_i.$$

Theory on Likelihood Bounds

The model is a convex optimization problem, and connects to many statistical theories:

- Statistical Divergence theory: provide a bound on KL divergence
- Bayesian Statistics with the threshold γ estimated by samples: confidence level on the true distribution
- Non-parametric Empirical Likelihood theory: inference based on empirical likelihood by Owen
- Asymptotic Theory of the likelihood region
- Possible extensions to deal with Continuous Case

Wang, Glynn and Y [2012,2016]

DRO using Wasserstein Ambiguity Set

By the Kantorovich-Rubinstein theorem, the Wasserstein distance between two distributions can be expressed as the minimum cost of moving one to the other, which is a semi-infinite transportation LP.

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Theorem

When using the Wasserstein ambiguity set

$$\mathcal{D}_N := \{F_\xi \mid P(\xi \in \Xi) = 1 \;\&\; d(F_\xi, \hat{F}_N) \leq \varepsilon_N\},$$

where $d(F_1, F_2)$ is the Wasserstein distance function and N is the sample size, the DRO model satisfies the following properties:

- *Finite sample guarantee : the correctness probability $\bar{P}^N$ is high*
- *Asymptotic guarantee : $\bar{P}^\infty(\lim_{N\to\infty} \hat{x}_{\varepsilon_N} = x^*) = 1$*
- *Tractability : DRO is in the same complexity class as SAA*

Mohajerin Esfahani & Kuhn [15, 17], Blanchet, Kang, Murthy [16], Duchi, Glynn, Namkoong

DRO for Logistic Regression

- Let $\{(\hat{\xi}_i, \hat{\lambda}_i)\}_{i=1}^N$ be a feature-label training set i.i.d. from P, and consider applying logistic regression :

$$\min_x \frac{1}{N}\sum_{i=1}^{N} \ell(x, \hat{\xi}_i, \hat{\lambda}_i) \text{ where } \ell(x,\xi,\lambda) = \ln(1+\exp(-\lambda x^T \xi))$$

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$$\min_x \sup_{F \in \mathcal{D}_N} \mathbb{E}_F[\ell(x, \xi_i, \lambda_i)]$$

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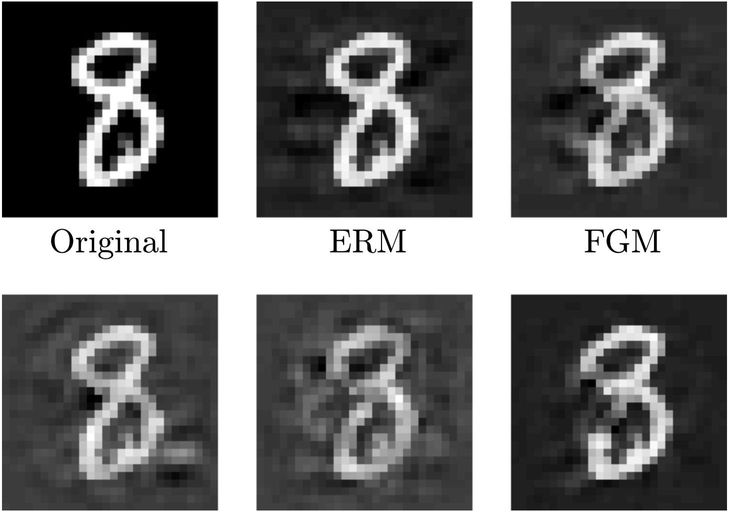
with the Wasserstein ambiguity set.

- When labels are considered to be error free, DRO with $\mathcal{D}_N$ reduces to regularized logistic regression:

$$\min_x \frac{1}{N}\sum_{i=1}^N \ell(x, \hat{\xi}_i, \hat{\lambda}_i) + \varepsilon\|x\|_*$$

Shafieezadeh Abadeh, Mohajerin Esfahani, & Kuhn, NIPS, [2015]

Results of the DRO Learning

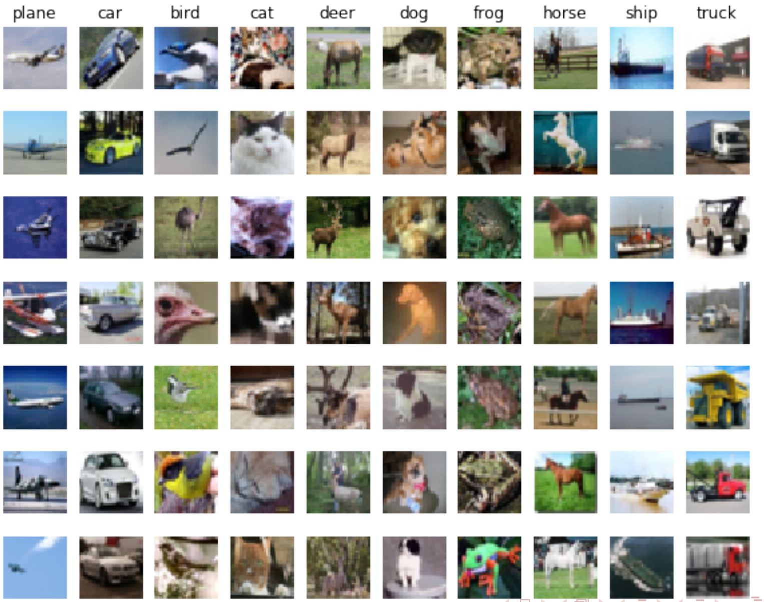


Original ERM FGM

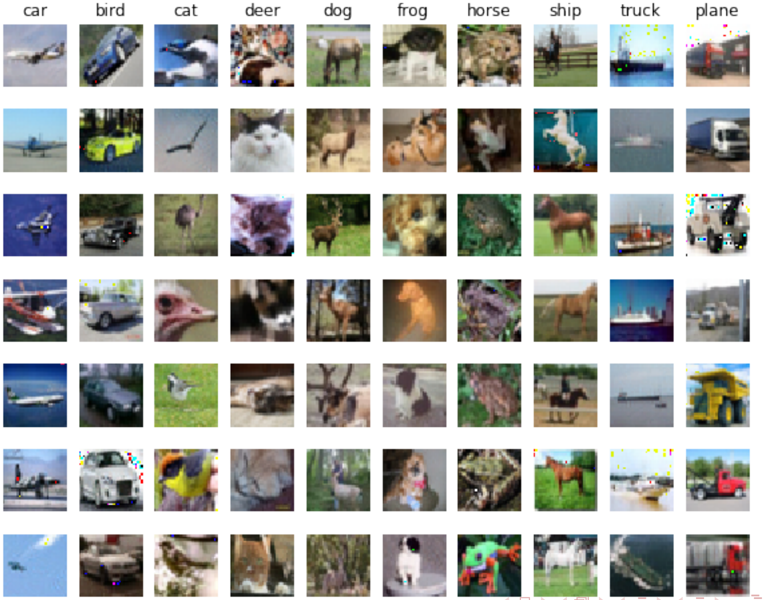
IFGM PGM WRM

Sinha, Namkoong and Duchi [2017]

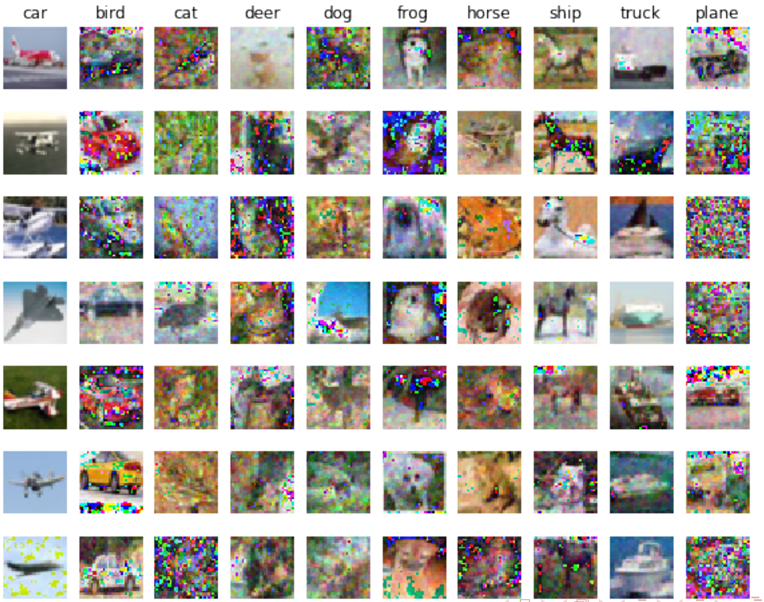
Results of the DRO Learning: Original



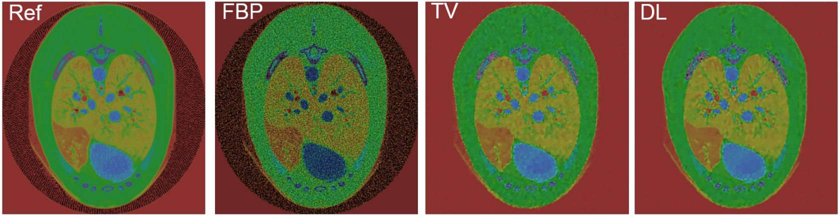
Results of the DRO Learning: Nonrobust



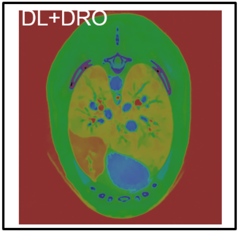
Results of the DRO Learning: DRO



Medical Application



Substantial noise reduction



Ref: Filtered Back Projection reconstructions of noise-free data
FBP: FBP reconstructions of noisy data
TV: TV-based reconstruction
DL: Dictionary Learning-based reconstruction
DL+DRO: DL+DRO to encourage low-rankness and robustness

Liu at all. [2017]

Summary of DRO under Moment, Likelihood or Wasserstein Ambiguity Set

- The DRO models yield a solution with a guaranteed confidence level to the possible distributions. Specifically, the confidence region of the distributions can be constructed upon the historical data and sample distributions.

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- The DRO models yield a solution with a guaranteed confidence level to the possible distributions. Specifically, the confidence region of the distributions can be constructed upon the historical data and sample distributions.
- The DRO models are tractable, and sometimes maintain the same computational complexity as the stochastic optimization models with known distribution.
- This approach can be applied to a wide range of problems, including inventory problems (e.g., newsvendor problem), portfolio selection problems, image reconstruction, machine learning, etc., with reported superior numerical results

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Planning under High-Dimensional Stochastic Data

Name	Symbol	Shares	Cost basis	Mkt value	Gain	Gain %
Apple Inc.	AAPL	1	122.4	86.29	-36.11	-29.5
Amazon.com, Inc.	AMZN	2	128.18	101.52	-26.66	-20.8
eBay Inc.	EBAY	1.25	32.23	17.45	-14.78	-45.85
Microsoft Corporation	MSFT	3.02	84.17	58.41	-25.76	-30.61
Google Inc.	GOOG	4	1733.4	1212.44	-521	-30.05
Yahoo! Inc.	YHOO	2	58.06	23.94	-34.12	-58.77
News Corporation	NWS	5	93.5	46.85	-46.65	-49.89
Time Warner Inc.	TWX	4	59.44	40.28	-19.16	-32.23
The Walt Disney Company	DIS	3	92.28	67.44	-24.84	-26.92
Bank of America Corporation	BAC	1	36.74	13.24	-23.5	-63.96
Best Buy Co., Inc.	BBY	1	39.87	27.75	-12.12	-30.4
Comcast Corporation	CMCSK	1	19.41	15.79	-3.62	-18.65
AT&T Inc.	T	2	70.02	56.46	-13.56	-19.37
Netflix, Inc.	NFLX	1	31.26	28.66	-2.6	-8.32
JPMorgan Chase & Co.	JPM	1	37.56	31.01	-6.55	-17.44
Dell Inc.	DELL	1	19.35	10.23	-9.12	-47.13
DivX, Inc.	DIVX	1	9.38	5.1	-4.28	-45.63
Wal-Mart Stores, Inc.	WMT	1	49.9	55.05	5.15	10.32
IAC/InterActiveCorp	IACI	1	19.53	15.77	-3.76	-19.25
Target Corporation	TGT	1	52	33.55	-18.45	-35.48
Adobe Systems Incorporated	ADBE	1	32.55	21.05	-11.5	-35.33
Electronic Arts Inc.	ERTS	1	46.31	15.34	-30.97	-66.88
Monster Worldwide, Inc.	MWW	1	24.75	11.55	-13.2	-53.33
Circuit City Stores, Inc.	CCTYQ	1	3.84	0.14	-3.71	-96.48
Gannett Co., Inc.	GCI	0.41	11.85	3.27	-8.58	-72.42
The McClatchy Company	MNI	0.14	1.24	0.11	-1.13	-91.21
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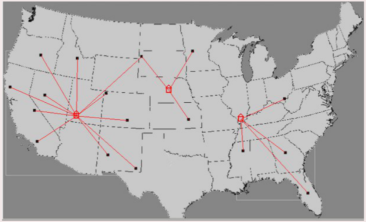
Portfolio Optimization



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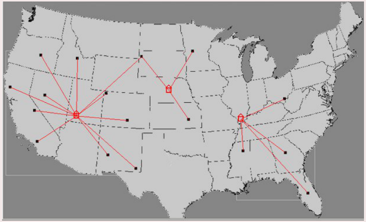

Portfolio Optimization **Facility Location**

$$\underset{x \in X}{\text{minimize}} \quad \mathbb{E}_p[f(x,\xi)]$$

where ξ is a high-dimensional random vector, and many possible return/demand high-dimensional joint distributions.

Price of Correlations

One can also consider the distributionally robust approach:

$$\underset{\mathbf{x}\in X}{\text{minimize}}\ \underset{p\in\mathcal{D}}{\text{maximize}}\ \mathbb{E}_p[f(\mathbf{x},\xi)]$$

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- We precisely quantify this risk as

  Price of Correlations (POC)

- We provide tight bounds on POC for various cost functions.

Price of Correlations

Define

- $\hat{\mathbf{x}}$ be the optimal solution of stochastic program with independent distribution $\hat{p}(\xi) = \prod_i p_i(\xi_i)$.

$$\boxed{\hat{\mathbf{x}} = \arg\min_{\mathbf{x}\in X} \mathbb{E}_{\hat{p}}[f(\mathbf{x},\xi)]}$$

- $\mathbf{x}^*$ be the optimal solution for the distributionally robust model.

$$\boxed{\mathbf{x}^* = \arg\min_{\mathbf{x}\in X} \max_{p\in\mathcal{D}} \mathbb{E}_p[f(\mathbf{x},\xi)]}$$

Then, Price of Correlations (POC), or Correlation Gap, is approximation ratio that $\hat{\mathbf{x}}$ achieves for distributionally robust model.

$$\mathrm{POC} = \frac{\max_{p\in\mathcal{D}} \ \mathbb{E}_p[f(\hat{\mathbf{x}},\xi)]}{\max_{p\in\mathcal{D}} \ \mathbb{E}_p[f(\mathbf{x}^*,\xi)]}$$

Price of Correlations

- Approximation of robust model
  - Minimax stochastic program can be replaced by stochastic program with independent distribution to get approximate solution.
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Question: What function class has large POC? What function class has small POC?

- Submodularity leads to small POC
- Supermodularity leads to large POC

Submodularity Leads to Small POC

- For any fixed x, function $f(\xi) = f(x, \xi)$ is submodular in random variable ξ
- Decreasing marginal cost, economies of scale

$$f(\max\{\xi, \theta\}) + f(\min\{\xi, \theta\}) \leq f(\xi) + f(\theta)$$

- For continuous functions: $\dfrac{\partial f(\xi)}{\partial \xi_i \partial \xi_j} \leq 0$

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Theorem

If $f(\cdot, \xi)$ is monotone and submodular in ξ, then POC $\leq e/(e-1)$.

Calinescu, Chekuri, Pál, Vondrák [2007] for binary random variables, Agrawal, Ding, Saberi, Y, [2010] for general domains

Supermodularity Leads to Large POC

- For any fixed x, function $f(\xi) = f(x, \xi)$ is supermodular in random variable ξ
- Increasing marginal cost

$$\frac{\partial f(\xi)}{\partial \xi_i \partial \xi_j} \geq 0$$

e.g., effects of increase in congestion as demand increases.

- In worst case distribution large values of one variable will appear with large values of other variable – highly correlated
- We show example of supermodular set function with $\mathrm{POC} = \Omega(2^n)$.

Agrawal, Ding, Saberi, Y, [2010]

Applications: Stochastic Bottleneck Matching

$$\boxed{\text{minimize}_{\mathbf{x}\in X} \;\; \mathrm{maximize}_{p\in\mathcal{P}} \;\; \mathbb{E}_p[\max_i \xi_i x_i]} \Rightarrow$$

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where expected value is under independent distribution $\hat{p}$.

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- Can be sampled efficiently, Chernoff type concentration bounds hold for monotone submodular functions.
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Beyond Submodularity?

Monotone Subadditive Functions?

- Preserves economy of scale
- Example with POC $\geq \Omega(\sqrt{n}/\log\log(n))$

Fractionally Subadditive?

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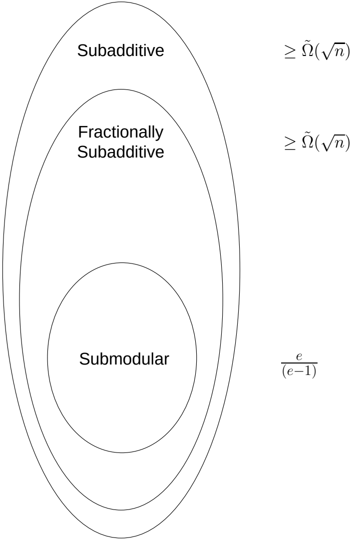

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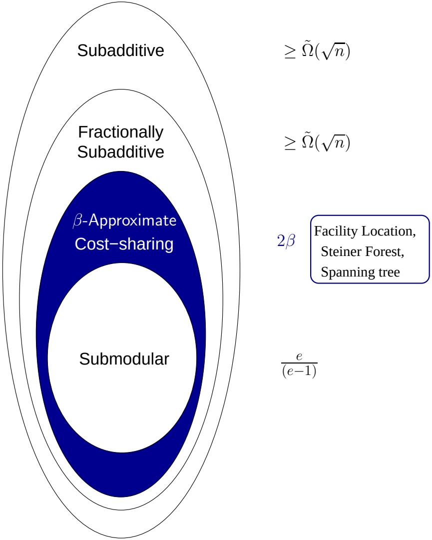
Cost-sharing to the rescue



Cross-Monotone Cost-Sharing

A cooperative game theory concept

- Can cost $f(\xi_1, \ldots, \xi_n)$ be charged to participants $1, \ldots, n$ so that the share charged to participant i decreases as the demands of other participants increase?
  [introduced by Thomson (1983, 1995) in context of bargaining]
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- β-approximate cost-sharing scheme: total cost charged is within β of the original (expected) function value

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Approximate cost-sharing schemes exist for non-submodular functions

- 3-approximate cost-sharing for facility location cost function
  [Pál, Tardos 2003]
- 2-approximate cost-sharing for Steiner forest cost function
  [Könemann, Leonardi, Schäfer 2005]

Bounding POC via Cost-Sharing

Theorem

If objective function $f(\cdot, \xi)$ is monotone in ξ with β-cost-sharing scheme, POC $\leq 2\beta$.

- POC ≤ 6 for two-stage stochastic facility location
- POC ≤ 4 for two-stage stochastic Steiner forest network design problem.

Agrawal, Ding, Saberi, Y, [2010]

The Cost-Sharing Condition is (near)-Tight

Theorem

If POC for function f is less than β, there exists a cross-monotone cost-sharing scheme with expected β-budget balance.

We show examples of

- Monotone submodular function with POC $\geq \frac{e}{e-1}$.
- Facility location with POC ≥ 3.
- Steiner tree network design with POC ≥ 2.

Agrawal, Ding, Saberi, Y, [2010]

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- Characterizes the risk associated with assuming independence in a stochastic optimization problem.
- Can be upper bounded using properties of objective function.

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Open questions

- Further characterizations of value of partial information in stochastic optimization problems
- Given partial information about correlations such as Covariance matrix
  - How does worst case distribution compare to maximum entropy distribution?
  - Block-wise independent distributions?

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- Modern applications include dynamic planning under uncertainty, reinforcement learning, social networking, and almost all other stochastic dynamic/sequential decision/game problems in Mathematical, Physical, Management and Social Sciences.

The Markov Decision/Game Process (continued)

- At each time step, the process is in some state $i \in \{1, ..., m\}$ and the decision maker chooses an action $j \in \mathcal{A}_i$ that is available in state i.

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The Optimal Cost-to-Go Value Vector I

Let $\mathbf{y} \in \mathbf{R}^m$ represent the cost-to-go values of the m states, ith entry for ith state, of a given policy. The MDP problem entails choosing the optimal value vector $\mathbf{y}^*$ such that it satisfies:

$$y_i^* = \min\{c_j + \gamma \mathbf{p}_j^T \mathbf{y}^*,\ \forall j \in \mathcal{A}_i\},\ \forall i,$$

with optimal policy

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In the Game setting, the conditions becomes:

$$y_i^* = \min\{c_j + \gamma \mathbf{p}_j^T \mathbf{y}^*, \ \forall j \in \mathcal{A}_i\}, \ \forall i \in I^-,$$

and

$$y_i^* = \max\{c_j + \gamma \mathbf{p}_j^T \mathbf{y}^*, \ \forall j \in \mathcal{A}_i\}, \ \forall i \in I^+.$$

They both are fix-point or saddle-point problems.

Value-Iteration (VI) Method

Let $\mathbf{y}^0 \in \mathbf{R}^m$ represent the initial cost-to-go values of the m states. The VI for MDP:

$$y_i^{k+1} = \min\{c_j + \gamma \mathbf{p}_j^T \mathbf{y}^k, \ \forall j \in \mathcal{A}_i\}, \ \forall i.$$

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The values inside the parenthesis are the so-called Q-values. Such operation can be written as

$$\mathbf{y}^{k+1} = T\mathbf{y}^k.$$

Randomized/Sample-Based Value-Iteration

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- Variance Reduction enables us to update the Q-values so that the needed number of samples is decreased from iteration to iteration.

Randomized/Sample-Based Value-Iteration Results

Two computation and sample complexity results are developed by (Sidford, Wang, Wu and Y [ICML 2017]) based on Variance Reduction (the VR technique has been used extensively in the design of fast stochastic methods for solving large–scale optimization problems in machine learning):

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Summary of MDP Value-Iteration and Near-Optimal Randomized Value-Iteration Result

[Sidford, Wang, Wu, Yang and Y NIPS 2018].

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- More recently, the result is substantially generalized to the Stochastic Game process.

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- Customers come and require a bundle of goods and bid for certain prices
- Decision: To sell or not to each individual customer?
- Objective: Maximize the revenue.

An Example

	Bid 1($t=1$)	Bid 2($t=2$)	.....	Inventory($\mathbf{b}$)
Price(π_t)	\$100	\$30	...	
Decision	x_1	x_2	...	
Pants	1	0	...	100
Shoes	1	0	...	50
T-shirts	0	1	...	500
Jackets	0	0	...	200
Hats	1	1	...	1000

Online Linear Programming Model

The classical offline version of the above program can be formulated as a linear (integer) program as all information data would have arrived: compute x_t, $t = 1, ..., n$ and

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Now we consider the online or streamline and data-driven version of this problem:

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- the bidder information is revealed sequentially along with the corresponding objective coefficient.
- an irrevocable decision must be made as soon as an order arrives without observing or knowing the future data.

Model Assumptions

Main Assumptions

- $0 \leq a_{it} \leq 1$, for all (i, t);
- $\pi_t \geq 0$ for all t
- The bids $(\mathbf{a}_t, \pi_t)$ arrive in a random order (rather than from some iid distribution).

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Denote the offline LP maximal value by $OPT(A, \pi)$. We call an online algorithm $\mathcal{A}$ to be c-competitive if and only if

$$E_\sigma \left[\sum_{t=1}^{n} \pi_t x_t(\sigma, \mathcal{A}) \right] \geq c \cdot OPT(A, \pi) \ \forall (A, \pi),$$

where σ is the permutation of arriving orders.

In what follows, we let

$$B = \min_i \{b_i\} (> 0).$$

Main Results: Necessary and Sufficient Conditions

Theorem

For any fixed $0 < \epsilon < 1$, *there is no online algorithm for solving the linear program with competitive ratio* $1 - \epsilon$ *if*

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- Consider $m = 1$ and inventory level B, one can construct an instance where $OPT = B$, and there will be a loss of $\sqrt{B}$ with a high probability, which give an approximation ratio $1 - \frac{1}{\sqrt{B}}$.

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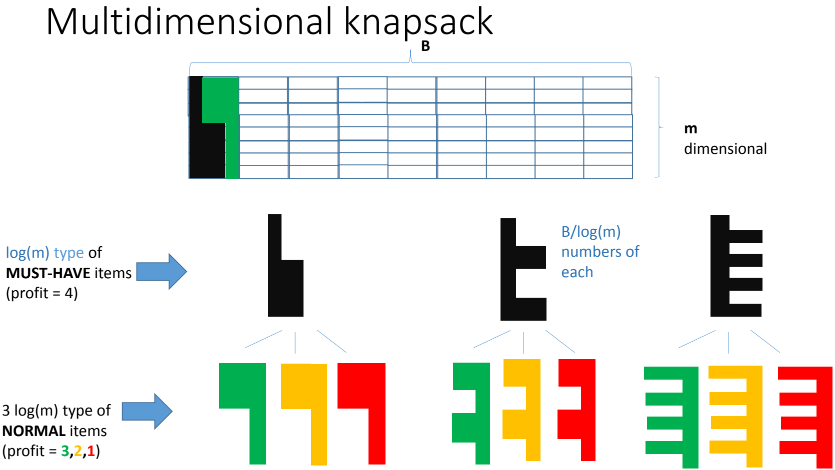
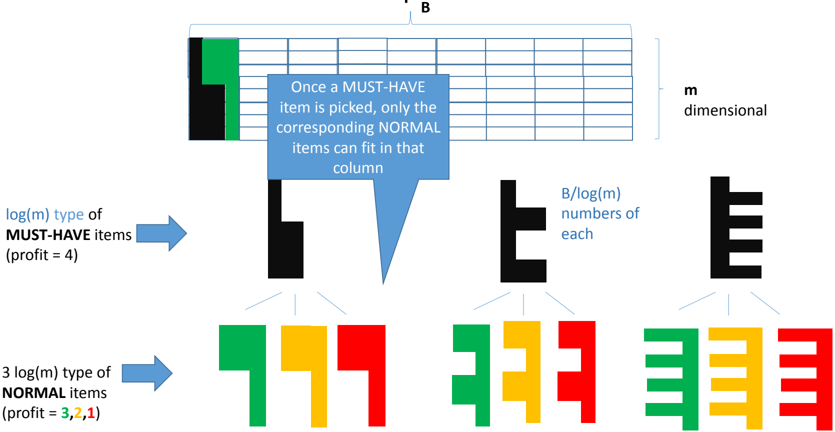
- Consider general m and inventory level B for each good. We are able to construct an instance to decompose the problem into $\log(m)$ separable problems, each of which has an inventory level $B/\log(m)$ on a composite "single good" and $OPT = B/\log(m)$.
- Then, with hight probability each "single good" case has a loss of $\sqrt{B/\log(m)}$ and the total loss of $\sqrt{B \cdot \log(m)}$. Thus, approximation ratio is at best $1 - \frac{\sqrt{\log(m)}}{\sqrt{B}}$.


Multidimensional knapsack
B
m
dimensional
log(m) type of
MUST-HAVE items
(profit = 4)
B/log(m)
numbers of
each
3 log(m) type of
NORMAL items
(profit = 3,2,1)

Necessary Result II

Multidimensional knapsack



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- Unlike dynamic programming, the decision maker does not have full information/data so that a backward recursion can not be carried out to find an optimal sequential decision policy.
- Thus, the online algorithm needs to be learning-based, in particular, learning-while-doing.

Price Observation of Online Learning I

The problem would be easy if there are "ideal prices":

	Bid 1($t=1$)	Bid 2($t=2$)	.....	Inventory($\mathbf{b}$)	$\mathbf{p}^*$
Bid(π_t)	\$100	\$30	...		
Decision	x_1	x_2	...		
Pants	1	0	...	100	\$45
Shoes	1	0	...	50	\$45
T-shirts	0	1	...	500	\$10
Jackets	0	0	...	200	\$55
Hats	1	1	...	1000	\$15

Price Observation of Online Learning II

- Pricing the bid: The optimal dual price vector $\mathbf{p}^*$ of the offline LP problem can play such a role, that is $x_t^* = 1$ if $\pi_t > \mathbf{a}_t^T \mathbf{p}^*$ and $x_t^* = 0$ otherwise, yields a near-optimal solution.

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- One-time learning algorithm: learn the price vector once using the initial ϵn input.
- Dynamic learning algorithm: dynamically update the prices at a carefully chosen pace.

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  and get the optimal dual solution $\hat{\mathbf{p}}$;
- Determine the future allocation x_t as:

$$x_t = \begin{cases} 0 & \text{if } \pi_t \leq \hat{\mathbf{p}}^T \mathbf{a}_t \\ 1 & \text{if } \pi_t > \hat{\mathbf{p}}^T \mathbf{a}_t \end{cases}$$

  as long as $a_{it} x_t \leq b_i - \sum_{j=1}^{t-1} a_{ij} x_j$ for all i; otherwise, set $x_t = 0$.

One-Time Learning Algorithm Result

Theorem

For any fixed $\epsilon > 0$, the one-time learning algorithm is $(1-\epsilon)$ competitive for solving the linear program when

$$B \geq \Omega\left(\frac{m \log(n/\epsilon)}{\epsilon^3}\right)$$

This is one ϵ worse than the optimal bound.

Outline of the Proof

- With high probability, we clear the market;
- With high probability, the revenue is near-optimal if we include the initial ϵ portion revenue;
- With high probability, the first ϵ portion revenue, a learning cost, doesn't contribute too much.

Then, we prove that the one-time learning algorithm is $(1-\epsilon)$ competitive under condition $B \geq \frac{6m\log(n/\epsilon)}{\epsilon^3}$.

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At time $\ell \in \{\epsilon n, 2\epsilon n, \ldots\}$, the price vector is the optimal dual solution to the following linear program:

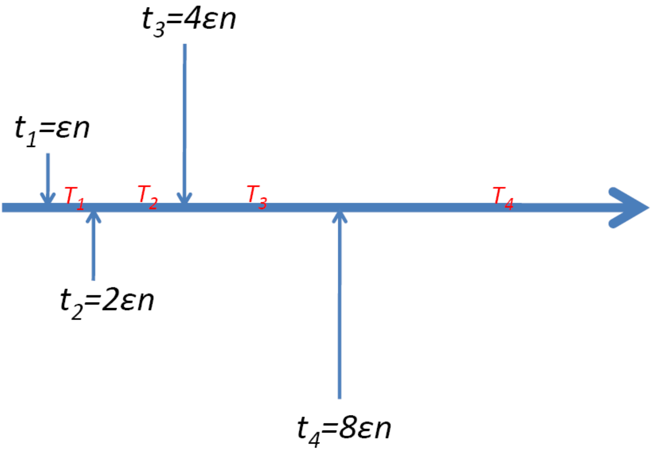
$$\begin{array}{lll} \text{maximize}_{\mathbf{x}} & \sum_{t=1}^{\ell} \pi_t x_t & \\ \text{subject to} & \sum_{t=1}^{\ell} a_{it} x_t \leq (1-h_\ell)\frac{\ell}{n} b_i & i = 1, \ldots, m \\ & 0 \leq x_t \leq 1 & t = 1, \ldots, \ell \end{array}$$

where

$$h_\ell = \epsilon\sqrt{\frac{n}{\ell}};$$

and this price vector is used to determine the allocation for the next immediate period.

Geometric Pace/Grid of Price Updating



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- The numbers h_ℓ play an important role in improving the condition on B in the main theorem. It basically balances the probability that the inventory ever gets violated and the lost of revenue due to the factor $1 - h_\ell$.
- Choosing large h_ℓ (more conservative) at the beginning periods and smaller h_ℓ (more aggressive) at the later periods, one can now control the loss of revenue by an ϵ order while the required size of B can be weakened by an ϵ factor.

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Gupta/Molinaro [2014]	$B \geq \frac{\log m}{\epsilon^2}$	Dynamic*
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Table: Comparison of several existing results

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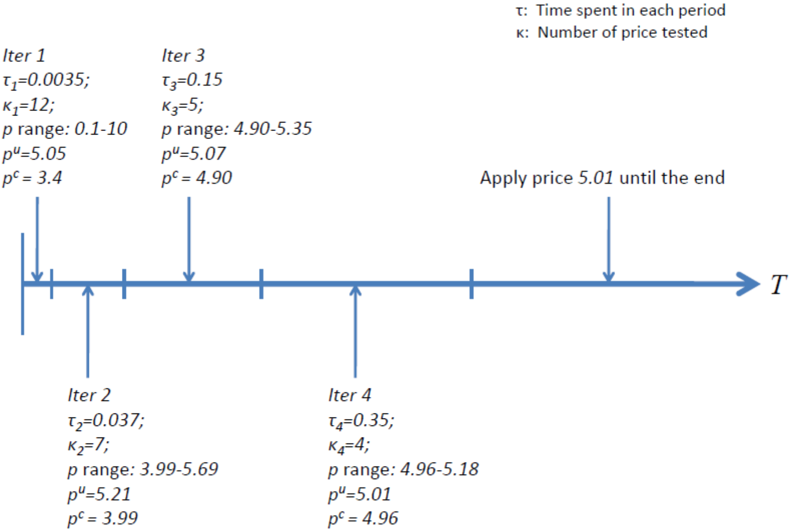
- Demand arrives in a Poisson process, where the arrival rate $\lambda(p)$ depends only on the instantaneous price posted by the seller.
- Objective is to maximize the expected revenue.
- Near optimal algorithm found for the one good case (Wang, Deng and Y [2014]).

Geometric Pace of Price Testing



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